GAUSSIAN PROCESSES
EXERCISE SHEET 1: STABLE LAWS

Exercise 1 (Symmetric α-stable laws)**.** *Prove that the symmetric α-stable laws (those with characteristic functions $\mathbb{E}[\exp(i\lambda X)] = \exp(-C|\lambda|^\alpha)$ for $0 < \alpha \le 2$ and $C > 0$) are indeed strictly stable laws. What is b_n in each case?*

Exercise 2 (Infinitely divisible laws)**.** *Another class of laws that Gaussian is part of are called infinitely divisible laws: a probability law μ is infinitely divisible, if for all $n \in \mathbb{N}$ we can find $X_1, \dots, X_n$ i.i.d. (but not necessarily distributed like μ) such that $X_1 + \cdots + X_n \sim \mu$.*

(1) Show that all stable laws are infinitely divisible.

(2) Find a probability law that is not stable but is infinitely divisible. Can you find one with finite variance?

Exercise 3. *Verify the stability of the standard Cauchy distribution by its distribution function $p(x) = \frac{1}{\pi(1+x^2)}$. Provide its characteristic function and determine b_n.*

Exercise 4 (An alternative definition of strictly stable laws)**.** *Use the result of Exercise 7 to deduce that if μ is a strictly stable law, then the following condition is satisfied:*

- *for all $a, b > 0$ there is some $c > 0$ such that if $X_1, X_2 \sim \mu$ are i.i.d., then $aX_1 + bX_2 \sim cX$, where $X \sim \mu$ as well.*

Exercise 5 (Discrete strictly stable laws)**.** *Are there any discrete strictly stable laws? Explain why or why not. You can use the hint of Exercise 7.*

Exercise 6 (About $\alpha \le 2$ (Not Examinable))**.** *Prove that for $\alpha > 2$ and $C > 0$, $\mathbb{E}\exp(i\lambda X) = \exp(-C|\lambda|^\alpha)$ cannot be the characteristic function of a real-valued random variable.*
(Hint: Check the second moment.)

Exercise 7 (The value of b_n (Not Examinable))**.** *Prove that for any strictly stable law $b_n = n^{1/\alpha}$ for some $\alpha \in (0, 2]$. (Hint: Solve the functional equation for the characteristic function. To regularize the characteristic function, one may consider $Z := X - X'$, where X' is an independent copy of X. Then Z has a symmetric α-stable law.)*